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Fluctuations in the random-field Ising model

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Abstract. Fluctuations can break down mean-field critical behaviour. For the random-field Ising model, fluctuations caused by the randomness are important. The Ginzburg criteria based on two kinds of mean-field theory are discussed. One is the mean-field theory using the replica method and the other is the site-dependent mean-field theory. An argument that justifies the use of the site-dependent mean-field theory to obtain critical properties for the random-field Ising model is given.

1. Introduction

Although the critical properties of the random-field Ising model (RFIM) have been extensively studied for a long time [1], they are not fully understood. Fluctuations can alter the mean-field critical behaviour. Both thermal and random fluctuations are present in the RFIM.

The Ginzburg criterion [2] has been used to estimate the size of the critical regions. For the RFIM, the criterion was discussed in [3]. Random-field fluctuations are shown to be important below the upper critical dimension $d_u = 6$.

There has been an attempt to extract the critical behaviour from a mean-field theory (MFT) [4]. The theory used is site dependent and fluctuations induced by the randomness are included in the MFT. Critical exponents are obtained on the assumption that thermal fluctuations are irrelevant.

In this article, we give an argument supporting the validity of using the site-dependent MFT. It will be shown that the remaining fluctuations seem to be irrelevant to the critical properties. To make a comparison, a MFT based on the replica method is also discussed. The Ginzburg criterion in this approach is reconsidered and its relevance to the possible replica symmetry breaking in the RFIM is discussed.

2. Fluctuations in the site-dependent MFT

The RFIM is represented by the following Hamiltonian:

$$\mathcal{H} = -\frac{1}{2}\sum_{r,r'} J(r-r')S(r)S(r') - \sum_{r} h(r)S(r) \tag{1}$$

where $J(r)$ is short-range ferromagnetic interaction and the Ising spin $S(r)$ is located at lattice sites. The independent random fields $h(r)$ are assumed to obey the Gaussian distribution

$$P(h(r)) = \frac{1}{\sqrt{2\pi H^2}} \exp\left(-\frac{h(r)^2}{2H^2}\right). \tag{2}$$

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First we will derive the site-dependent MFT by the saddle point method. The partition function is given by

$$\begin{aligned} Z &= \mathrm{Tr}\, \mathrm{e}^{-\beta\mathcal{H}} \\ &= \int \prod_r \frac{\mathrm{d}X(r)}{\sqrt{2\pi}} \; \exp\left(-\frac{1}{2}\sum_{r,r'} X(r)\tilde{J}^{-1}(r-r')X(r')\right) \\ &\quad \times \sum_{\{S(r)\}} \exp\left(\sum_r X(r)S(r) + \sum_r \tilde{h}(r)S(r)\right) \\ &= \int \prod_r \frac{\mathrm{d}X(r)}{\sqrt{2\pi}} \; \exp\left\{-\frac{1}{2}\sum_{r,r'} X(r)\tilde{J}^{-1}(r-r')X(r') \right. \\ &\quad \left. + \sum_r \ln\left[2\cosh\left(X(r)+\tilde{h}(r)\right)\right]\right\} \end{aligned} \tag{3}$$

where we have used $\tilde{J} \equiv \beta J$ and $\tilde{h} \equiv \beta h$. The magnetization is expressed as

$$\begin{aligned} M(r'') &= \langle S(r'')\rangle \\ &= \frac{1}{Z}\int \prod_r \frac{\mathrm{d}X(r)}{\sqrt{2\pi}} \; \exp\left(-\frac{1}{2}\sum_{r,r'} X(r)\tilde{J}^{-1}(r-r')X(r')\right) \\ &\quad \times \sum_{\{S(r)\}} S(r'') \exp\left(\sum_r X(r)S(r) + \sum_r \tilde{h}(r)S(r)\right) \\ &= \frac{1}{Z}\int \prod_r \frac{\mathrm{d}X(r)}{\sqrt{2\pi}} \; \sum_{r'} \tilde{J}^{-1}(r''-r')X(r') \\ &\quad \times \exp\left(-\frac{1}{2}\sum_{r,r'} X(r)\tilde{J}^{-1}(r-r')X(r')\right) \\ &\quad \times \sum_{\{S(r)\}} \exp\left(\sum_r X(r)S(r) + \sum_r \tilde{h}(r)S(r)\right) \end{aligned} \tag{4}$$

where $\langle\cdots\rangle$ is the thermal average. It is natural to define a field $\varphi(r)$ as

$$\varphi(r) \equiv \sum_{r'} \tilde{J}^{-1}(r-r')X(r'). \tag{5}$$

Then the thermal average of the field $\varphi(r)$ is just the magnetization $M(r)$. Using the field $\varphi(r)$, the partition function is written as

$$\begin{aligned} Z &= \int \{\mathrm{d}\varphi(r)\} \; \exp\left\{-\frac{1}{2}\sum_{r,r'} \varphi(r)\tilde{J}(r-r')\varphi(r') \right. \\ &\quad \left. + \sum_r \ln\left[2\cosh\left(\sum_{r'} \tilde{J}(r-r')\varphi(r') + \tilde{h}(r)\right)\right]\right\}. \end{aligned} \tag{6}$$

The mean-field equations are obtained from the saddle points of equation (6):

$$M^0(r) = \tanh\left(\sum_{r'} \tilde{J}(r-r')M^0(r') + \tilde{h}(r)\right). \tag{7}$$

These equations have many solutions near the phase transition line. The solutions will be labelled by α. The mean-field partition function may be written as

$$Z_{MF} = \sum_\alpha \exp\left\{-\frac{1}{2}\sum_{r,r'} M^{0,\alpha}(r)\tilde{J}(r-r')M^{0,\alpha}(r') + \sum_r \ln\left[2\cosh\left(\sum_{r'}\tilde{J}(r-r')M^{0,\alpha}(r') + \tilde{h}(r)\right)\right]\right\} = \sum_\alpha \exp(-\beta F^\alpha) \tag{8}$$

where F^α is used as free energy for one particular solution. Within the MFT, the magnetization may be given by

$$\langle S(r)\rangle = \frac{\partial \ln Z_{MF}}{\beta\,\partial h(r)} = \sum_\alpha M^{0,\alpha}(r) w^\alpha \tag{9}$$

where the weight of a solution is defined as

$$w^\alpha \equiv \frac{\exp(-\beta F^\alpha)}{Z_{MF}}. \tag{10}$$

Connected correlation functions may be given by

$$\langle S(r)S(r')\rangle_c = \frac{\partial^2 \ln Z_{MF}}{\beta^2\,\partial h(r)\,\partial h(r')} = \frac{1}{\beta}\sum_\alpha w^\alpha g^\alpha_{r,r'} + \sum_\alpha M^{0,\alpha}(r)M^{0,\alpha}(r')w^\alpha - \sum_\alpha M^{0,\alpha}(r)w^\alpha \sum_\gamma M^{0,\gamma}(r')w^\gamma \tag{11}$$

where

$$g^\alpha_{r,r'} \equiv \frac{\partial M^{0,\alpha}(r)}{\partial h(r')}. \tag{12}$$

Equations (7) to (12) are the same as those used in reference [4].

To discuss the Ginzburg criterion for the RFIM, the following quantity will be considered:

$$G \equiv \left(\int_V \mathrm{d}\boldsymbol{r}\,\left\langle(\varphi(\boldsymbol{r}) - \langle\varphi(\boldsymbol{r})\rangle)(\varphi(\boldsymbol{0}) - \langle\varphi(\boldsymbol{0})\rangle)\right\rangle\right) \Big/ \int_V \mathrm{d}\boldsymbol{r}\,\left(\langle\varphi(\boldsymbol{r})\rangle\right)^2 \tag{13}$$

where V is the correlation volume. The numerator, which represents fluctuations, is estimated from

$$\int_V \mathrm{d}\boldsymbol{r}\,\left\langle(\varphi(\boldsymbol{r}) - \langle\varphi(\boldsymbol{r})\rangle)(\varphi(\boldsymbol{0}) - \langle\varphi(\boldsymbol{0})\rangle)\right\rangle \approx kT\chi_T \tag{14}$$

where χ_T is the susceptibility. Its singularity near the critical point is given by

$$\chi_T \sim (T - T_c)^{-\gamma} \tag{15}$$

where γ is the exponent obtained from the site-dependent MFT [4]. The site-dependent magnetization near the critical point is

$$\langle\varphi(r)\rangle \sim \mathrm{O}(1) \tag{16}$$

for a non-zero random field. Therefore the denominator of (13) is estimated as

$$\xi^d \sim (T - T_c)^{-\nu d} \tag{17}$$

near the critical point and ξ is the correlation length. Using (15) and (17),

$$G \sim (T - T_c)^{\nu(d-2+\eta)} \tag{18}$$

where the scaling relation $\gamma = \nu(2-\eta)$ has been used. This implies that the fluctuations do not seem to change the critical behaviour when

$$d > 2 - \eta. \tag{19}$$

In particular it is plausible that the site-dependent MFT gives correct critical properties for $d = 3$.

3. Fluctuations in the replica MFT

In the replica MFT, the magnetization is usually assumed to be uniform. This can be naturally derived from a saddle point in the replica formalism. In the following, the MFT using a replica is reviewed first. Afterwards, fluctuations are discussed for this case.

The replicated partition function is

$$Z^n = \int \prod_\alpha \prod_r \frac{\mathrm{d}X^\alpha(r)}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sum_\alpha \sum_{r,r'} X^\alpha(r)\tilde{J}^{-1}(r-r')X^\alpha(r')\right) \times \sum_{\{S^\alpha(r)\}} \exp\left(\sum_\alpha \sum_r X^\alpha(r)S^\alpha(r) + \sum_r \tilde{h}(r)\sum_\alpha S^\alpha(r)\right) \tag{20}$$

where α represents replicas in this section. Taking the Gaussian random average, we have

$$\langle Z^n\rangle_h = \int \prod_r \frac{\mathrm{d}h(r)}{\sqrt{2\pi}} \int \{\mathrm{d}\varphi^\alpha(r)\} \exp\left\{-\frac{1}{2}\sum_\alpha \sum_{r,r'} \varphi^\alpha(r)\tilde{J}(r-r')\varphi^\alpha(r') - \frac{1}{2}\sum_r h(r)^2 + \sum_\alpha \sum_r \ln\left[2\cosh\left(\sum_{r'} \tilde{J}(r-r')\varphi^\alpha(r') + \beta H h(r)\right)\right]\right\} \tag{21}$$

where $\langle\cdots\rangle_h$ represents the random average and $\varphi^\alpha(r)$ is defined similarly to (5).

The mean-field equation is obtained from the saddle point:

$$-\sum_r \tilde{J}(r-r')M^0 + \left.\frac{\partial W_0}{\partial\varphi^\alpha(r')}\right|_{\varphi^\alpha(r')=M^0} = 0 \tag{22}$$

where $W_0(\varphi^\alpha(r))$ is defined by

$$\mathrm{e}^{W_0(\varphi^\alpha(r))} \equiv \int \prod_r \frac{\mathrm{d}h(r)}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\sum_r h(r)^2 + \sum_\alpha \sum_r \ln\left[2\cosh\left(\sum_{r'} \tilde{J}(r-r')\varphi^\alpha(r') + \beta H h(r)\right)\right]\right\}. \tag{23}$$

We have assumed replica-symmetric magnetization M^0. Taking the $n \to 0$ limit for the replica, the mean-field equation becomes

$$M^0(r) = \int \prod_r \frac{\mathrm{d}h(r)}{\sqrt{2\pi}} \mathrm{e}^{-(1/2)h(r)^2} \tanh\left(\sum_{r'} \tilde{J}(r-r')M^0(r') + \beta H h(r)\right). \tag{24}$$

The magnetic susceptibility χ is easily calculated within the MFT. Near the critical point it becomes

$$\chi \sim \frac{1}{T - T_c}. \tag{25}$$

This form shows the same singularity as the non-random ferromagnet. Taking into account the relation

$$\chi = \beta \sum_{r'} \left\langle \langle S(r)S(r')\rangle_c \right\rangle_h \tag{26}$$

the fluctuations are also of the same form as those for the homogeneous ferromagnet. Therefore the Ginzburg criterion seems to be essentially unchanged in the presence of the random fields, which contradicts the previous result [3]. What is wrong? To see the origin of this contradiction, the fluctuations will be calculated directly within the Gaussian approximation.

The connected correlation function is

$$\begin{aligned}\langle\langle S(r)S(r')\rangle_c\rangle_h &= \lim_{n\to 0}\frac{1}{n}\int\prod_\alpha\prod_{r''}\frac{\mathrm{d}X''^\alpha(r)}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\sum_\alpha\sum_{r'',r'''}X^\alpha(r'')\tilde{J}^{-1}(r''-r''')X^\alpha(r''')\right)\\ &\times\sum_{\{S^\alpha(r)\}}\left(\sum_\alpha S^\alpha(r)\right)\left(\sum_\beta S^\beta(r')\right)\\ &\times\exp\left(\sum_\alpha\sum_{r''}X^\alpha(r'')S^\alpha(r'')+\sum_{r''}\tilde{h}(r'')\sum_\alpha S^\alpha(r'')\right)\\ &= \lim_{n\to 0}\frac{1}{n}\sum_{r''}\sum_{r'''}\tilde{J}^{-1}(r-r'')\tilde{J}^{-1}(r'-r''')\sum_{\alpha,\beta}\langle X^\alpha(r'')X^\beta(r''')\rangle_n \qquad (27)\end{aligned}$$

where $\langle\cdots\rangle_n$ represents

$$\int\prod_\alpha\prod_r\frac{\mathrm{d}X^\alpha(r)}{\sqrt{2\pi}}\ \mathrm{e}^{-\mathcal{H}_n}\cdots$$

and

$$\mathcal{H}_n\equiv\frac{1}{2}\sum_\alpha\sum_{r,r'}X^\alpha(r)\tilde{J}^{-1}(r-r')X^\alpha(r')+\sum_r W_0(X^\alpha(r)). \qquad (28)$$

$W_0(X^\alpha(r))$ is defined by

$$\mathrm{e}^{-W_0(X^\alpha(r))}\equiv\int\frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}\sum_{\{S^\alpha(r)\}}\exp\left(\sum_\alpha X^\alpha(r)S^\alpha(r)+\tilde{h}(r)\sum_\alpha S^\alpha(r)\right). \qquad (29)$$

The field $X^\alpha(r)$ is written as

$$X^\alpha(r)=X^*+x^\alpha(r) \qquad (30)$$

where X^* is the saddle point and $x^\alpha(r)$ represents fluctuations. The Gaussian approximation corresponds to a truncation up to the quadratic terms in $x^\alpha(r)$. Equation (28) is approximated as

$$\begin{aligned}\mathcal{H}_n\simeq\mathcal{H}_n^{MF}&+\frac{1}{2}\sum_\alpha\sum_{r,r'}x^\alpha(r)\tilde{J}^{-1}(r-r')x^\alpha(r')\\ &+\frac{1}{2}\sum_{\alpha,\beta}\sum_r\left.\frac{\partial^2W_0(X^\alpha(r))}{\partial X^\alpha(r)\,\partial X^\beta(r)}\right|_{X^\alpha(r)=X^*}x^\alpha(r)x^\beta(r) \qquad (31)\end{aligned}$$

where

$$\mathcal{H}_n^{MF}=\frac{1}{2}nN\sum_r\tilde{J}^{-1}(r)(X^*)+NW_0(X^*). \qquad (32)$$

Here

$$W_0(X^*)\overset{n\to 0}{\longrightarrow}-n\int\frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}\ln[2\cosh(X^*+\tilde{h}(r))]. \qquad (33)$$

The second derivative in equation (31) is calculated to be

$$\left.\frac{\partial^2 W_0(X^\alpha(r))}{\partial X^\alpha(r)\,\partial X^\beta(r)}\right|_{X^\alpha(r)=X^*}$$
$$= -\delta_{\alpha\beta} - (1-\delta_{\alpha\beta})$$
$$\times\left(\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}[2\cosh(X^*+\tilde{h}(r))]^{n-2}[2\sinh(X^*+\tilde{h}(r))]^2\right)$$
$$\times\left(\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}[2\cosh(X^*+\tilde{h}(r))]^{n}\right)^{-1}$$
$$+\left\{\left(\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}[2\cosh(X^*+\tilde{h}(r))]^{n-1}[2\sinh(X^*+\tilde{h}(r))]\right)\right.$$
$$\left.\times\left(\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}[2\cosh(X^*+\tilde{h}(r))]^{n}\right)^{-1}\right\}^2$$
$$\equiv A^{\alpha\beta}. \tag{34}$$

Next the following Gaussian integral will be calculated:

$$\int\prod_\alpha\prod_r \frac{\mathrm{d}x^\alpha(r)}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\sum_\alpha\sum_{r,r'} x^\alpha(r)\tilde{J}^{-1}(r-r')x^\alpha(r') - \frac{1}{2}\sum_{\alpha,\beta}A^{\alpha\beta}\sum_r x^\alpha(r)x^\beta(r)\right). \tag{35}$$

Using Fourier transformation, it becomes

$$\int\{\mathrm{d}x^\alpha(\boldsymbol{q})\}\ \exp\left[-\frac{1}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d\left(\tilde{J}^{-1}(\boldsymbol{q})\boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{q})\cdot\boldsymbol{x}(\boldsymbol{q}) + \boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{q})\mathbf{A}\boldsymbol{x}(\boldsymbol{q})\right)\right]$$
$$= \exp\left\{-\frac{N}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d \ln\left[\det\left(\left[\mathbf{1}\Big/2\tilde{J}\sum_\mu\cos(q_\mu)\right]+\mathbf{A}\right)\right]\right\} \tag{36}$$

for the hypercubic lattice with nearest-neighbour interactions. The boldface notation has been used for a vector and a matrix in the replica space. As is seen from (34), the diagonal and off-diagonal elements of $\mathbf{A}$ can be abbreviated as a and b respectively. Using a and b, equation (36) becomes

$$\exp\left\{-\frac{N}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d \ln\left[\left(\left[1\Big/2\tilde{J}\sum_\mu\cos(q_\mu)\right]+a-b\right)^{n-1}\right.\right.$$
$$\left.\left.\times\left(\left[1\Big/2\tilde{J}\sum_\mu\cos(q_\mu)\right]+a-(1-n)b\right)\right]\right\}$$
$$\xrightarrow{n\to0}\ \exp\left\{-\frac{nN}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d\left[\ln\left(\left[1\Big/2\tilde{J}\sum_\mu\cos(q_\mu)\right]+a_0-b_0\right)\right.\right.$$
$$\left.\left.+\ b_0\Big/\left(\left[1\Big/2\tilde{J}\sum_\mu\cos(q_\mu)\right]+a_0-b_0\right)\right]\right\} \tag{37}$$

where a_0 and b_0 are obtained from the lowest-order terms of a and b, expanding in n. Their explicit forms are given by

$$a_0 = -1 + \left[\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))}\tanh(X^*+\tilde{h}(r))\right]^2 \tag{38}$$

and

$$b_0 = -\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))} \tanh^2(X^* + \tilde{h}(r)) + \left[\int \frac{\mathrm{d}h(r)}{\sqrt{2\pi H^2}}\ \mathrm{e}^{-(h(r)^2/(2H^2))} \tanh(X^* + \tilde{h}(r))\right]^2. \tag{39}$$

The average with respect to $\mathcal{H}_n$ is approximated as

$$\langle\cdots\rangle_n \simeq \mathrm{e}^{-\mathcal{H}_n^{MF}} \int \{\mathrm{d}x^\alpha(\boldsymbol{q})\}\ \exp\left[-\frac{N}{2}\sum_{\alpha,\beta}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d x^\alpha(\boldsymbol{q})\Lambda^{\alpha\beta}(\boldsymbol{q})x^\beta(-\boldsymbol{q})\right]\cdots \tag{40}$$

where the notation $\Lambda^{\alpha\beta}$ for the matrix is introduced. Using this expression, we have

$$\frac{1}{N}\sum_{r,r'}\langle x^\alpha(r)x^\beta(r')\rangle_n = \frac{1}{N}\langle x^\alpha(\boldsymbol{q}=\boldsymbol{0})x^\beta(\boldsymbol{q}=\boldsymbol{0})\rangle_n = -\frac{2\,\partial\langle 1\rangle_n}{N\,\partial\Lambda^{\alpha\beta}(\boldsymbol{q}=\boldsymbol{0})} \tag{41}$$

where

$$\langle 1\rangle_n = \mathrm{e}^{-\mathcal{H}_n^{MF}} \exp\left[-\frac{N}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d \ln(\det\boldsymbol{\Lambda}(\boldsymbol{q}))\right]. \tag{42}$$

The derivative becomes

$$\frac{\partial\langle 1\rangle_n}{\partial\Lambda^{\alpha\beta}(\boldsymbol{q}=\boldsymbol{0})} = \mathrm{e}^{-\mathcal{H}_n^{MF}} \exp\left[-\frac{N}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d \ln(\det\boldsymbol{\Lambda}(\boldsymbol{q}))\right]\left(-\frac{N}{2}\right)\frac{\tilde{\Lambda}^{\alpha\beta}(\boldsymbol{q}=\boldsymbol{0})}{\det\boldsymbol{\Lambda}(\boldsymbol{q}=\boldsymbol{0})} \tag{43}$$

where $\tilde{\Lambda}^{\alpha\beta}$ is the cofactor of the matrix $\boldsymbol{\Lambda}$. Using (43), equation (41) becomes

$$\frac{1}{N}\sum_{r,r'}\langle x^\alpha(r)x^\beta(r')\rangle_n = \mathrm{e}^{-\mathcal{H}_n^{MF}} \exp\left[-\frac{N}{2}\int\left(\frac{\mathrm{d}q}{2\pi}\right)^d \ln(\det\boldsymbol{\Lambda}(\boldsymbol{q}))\right](\boldsymbol{\Lambda}^{-1}(\boldsymbol{q}=\boldsymbol{0}))^{\beta\alpha}. \tag{44}$$

The diagonal and off-diagonal elements of the inverse matrix $\boldsymbol{\Lambda}^{-1}(\boldsymbol{q})$ are given by

$$c = \frac{-[b(n-2)+a+\tilde{J}^{-1}(\boldsymbol{q})]}{[b(n-1)+a+\tilde{J}^{-1}(\boldsymbol{q})][b-a-\tilde{J}^{-1}(\boldsymbol{q})]} \tag{45}$$

and

$$d = \frac{b}{[b(n-1)+a+\tilde{J}^{-1}(\boldsymbol{q})][b-a-\tilde{J}^{-1}(\boldsymbol{q})]} \tag{46}$$

respectively. Using these results, the fluctuations in the Gaussian approximation become

$$\begin{aligned}
&\frac{1}{N}\sum_{r,r'}\left\langle\langle S(r)S(r')\rangle_c\right\rangle_h \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \\
&\quad = \lim_{n\to 0}\frac{1}{nN}(\tilde{J}^{-1}(\boldsymbol{q}))^2\sum_{\alpha,\beta}\sum_{r'',r'''}\langle x^\alpha(r'')x^\beta(r''')\rangle_n \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}''-\boldsymbol{r}''')} \\
&\quad = (\tilde{J}^{-1}(\boldsymbol{q}))^2\left[\frac{a_0+\tilde{J}^{-1}(\boldsymbol{q})-2b_0}{(a_0+\tilde{J}^{-1}(\boldsymbol{q})-b_0)^2} + \frac{b_0}{(a_0+\tilde{J}^{-1}(\boldsymbol{q})-b_0)^2}\right] \\
&\quad = (\tilde{J}^{-1}(\boldsymbol{q}))^2\frac{1}{a_0+\tilde{J}^{-1}(\boldsymbol{q})-b_0}.
\end{aligned} \tag{47}$$

We see that the stronger singularities in the diagonal and off-diagonal elements are cancelled out and a weaker singularity survives. This singularity is of the same form as that of the corresponding homogeneous system.

The Ginzburg criterion is also obtained from the specific heat. The specific heat is easily calculated by using (37) in the Gaussian approximation. It diverges for $d \leqslant 6$. Therefore the correct Ginzburg criterion [3] is obtained in this case. The critical exponent α for the specific heat is given by

$$\alpha = 3 - \frac{d}{2} \tag{48}$$

in this approximation. This result satisfies the hyperscaling relation with the dimensional reduction by 2:

$$\alpha = 2 - \nu(d-2) \tag{49}$$

where $\nu = \frac{1}{2}$ in the Gaussian approximation. The dimensional reduction is a consequence of the replica symmetry. The weaker singularity in (47) is also a consequence of the replica-symmetric correlation function $\langle x^\alpha(r)x^\beta(r')\rangle_n$. This quantity has been shown to be replica asymmetric in reference [5] at the ferromagnetic transition point within the self-consistent screening approximation. The replica symmetry breaking is expected to resolve the problem of the cancellation of the singularities in the fluctuations. In section 2, many solutions are considered, which may be related to the replica symmetry breaking. This is why the discussion in section 2 for the fluctuations derived from the susceptibility may be justified.

4. Conclusion

The Ginzburg criterion for the RFIM is discussed by obtaining the fluctuations for two kinds of MFT. The fluctuations have been obtained from the susceptibility at the mean-field level. The mean-field susceptibility is essentially the same as the fluctuation that is constituted from connected correlation functions in the Gaussian approximation.

In the replica MFT, the singularity of the susceptibility is weaker than that expected from the specific heat. Closer inspection of the connected correlation functions reveals that stronger singularities are cancelled out because of the replica-symmetric treatment.

On the other hand, many solutions are included in the site-dependent MFT, which seems to correspond to the replica symmetry breaking. Therefore the mean-field susceptibility is expected to include fluctuations that lead to the correct derivation of the Ginzburg criterion. This is consistent with the fact that critical behaviour seems to be obtained correctly from the site-dependent MFT.

References

[1] Belanger D P and Young A P 1991 *J. Magn. Magn. Mater.* **100** 272
[2] Ginzburg V L 1960 *Fiz. Tverd. Tela* **2** 2031 (Engl. Transl. 1961 *Sov. Phys.–Solid State* **2** 1824)
[3] Kaufman M and Kardar M 1985 *Phys. Rev.* B **31** 2913
[4] Lancaster D, Marinari E and Parisi G 1995 *J. Phys. A: Math. Gen.* **28** 3959
[5] Mézard M and Young A P 1992 *Europhys. Lett.* **18** 653